ON SEMI-INFINITE COHOMOLOGY OF FINITE DIMENSIONAL GRADED ALGEBRAS

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Abstract. We describe a general setting for the definition of semi-infinite cohomology of finite dimensional algebras, and provide an interpretation of such cohomology in terms of derived categories. We apply this interpretation to compute semi-infinite cohomology of some modules over the small group at a root of unity, generalizing an earlier result of S. Arkhipov (conjectured by B. Feigin).

1. Introduction

Semi-infinite cohomology of associative algebras was studied by S. Arkhipov in [Ar1], [Ar2], [Ar3]; see also [S] (these works are partly based on an earlier paper by A. Voronov [V] where the corresponding constructions were introduced in the context of Lie algebras).

Recall that the definition of semi-infinite cohomology (see e.g. [Ar1], Definition 3.3.6) works in the following set-up. We are given an associative graded algebra A, two subalgebras N, $B \subset A$ such that $A = N \otimes B$ as a vector space, satisfying some additional assumptions. In this situation the space of semi-infinite Ext's, $Ext^{\infty/2+\bullet}(X, Y)$ is defined for X, Y in the appropriate derived categories. The definition makes use of explicit complexes (a version of the bar resolution). The aim of this note is to show that, at least under certain simplifying assumptions, $Ext^{\infty/2+\bullet}(X, Y)$ is a particular case of a general categorical construction.

To describe the situation in more detail, recall that starting from an algebra $A = N \otimes B$ as above, one can define another algebra $A^{\#}$, which also contains subalgebras identified with N, B, so that $A^{\#} = B \otimes N$. The semi-infinite Ext's, $Ext^{\infty/2+\bullet}(X, Y)$ are then defined for $X \in D(A^{\#} - mod)$, $Y \in D(A - mod)$, where $D(A^{\#} - mod)$, $D(A - mod)$ are derived categories of modules with certain restrictions on the grading.

Our categorical interpretation relies on the following construction. Given small categories $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ with functors $\Phi : \mathcal{B} \to \mathcal{A}$, $\Phi' : \mathcal{B} \to \mathcal{A}'$ one can define for $X \in \mathcal{A}$, $Y \in \mathcal{A}'$ the set of "morphisms from X to Y through $\mathcal{B}$"; we denote this set by $Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y)$. We then show that if $\mathcal{A} = D^b(A^{\#} - mod)$, $\mathcal{A}' = D^b(A - mod)$, and $\mathcal{B}$ is the full triangulated subcategory in $\mathcal{A}$ generated by N-injective $A^{\#}$-modules, then, $\mathcal{B}$ is identified with a full subcategory in $\mathcal{A}'$ generated by N-projective A-modules, and, under certain assumptions, one has

$$Ext^{\infty/2+i}(X, Y) = Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y[i]). \tag{1}$$

Notice that description (1) of $Ext^{\infty/2+i}(X, Y)$ is "internal" in the derived category, i.e. refers only to the derived categories and their full subcategories rather than to a particular category of complexes.

An example of the situation considered in this paper is provided by a small quantum group at a root of unity [L], or by the restricted enveloping algebra of a simple Lie algebra in positive characteristic. Computation of semi-infinite cohomology in the former case is due to S. Arkhipov [Ar1] (the answer suggested as a conjecture by B. Feigin). An attempt to find a natural interpretation of this answer was the starting point for the present work. In section 6 we sketch a generalization of Arkhipov's Theorem based on our description of semi-infinite cohomology and the results of [ABG], [BL]. Similarly, the main result of [B1] yields a description of semi-infinite cohomology of tilting modules over the "big" quantum group restricted to the small quantum group as cohomology with support of coherent IC sheaves on the nilpotent cone [B2].

It should be noted that some definitions of semi-infinite cohomology found in the literature apply in a more general (or different) situation than the one considered in the present paper. An important example is provided by affine Lie algebras; in fact, semi-infinite cohomology has first been defined in this context, related to the physical notion of BRST reduction. We hope that our approach can be extended to such more general setting. Some of the ingredients needed for the generalization are provided by [P].

The paper is organized as follows. Section 2 is devoted to basic general facts about "Hom through a category". Section 3 contains the definition of the algebra $A^{\#}$ and its propeties. In section 4 we recall the definition of semi-infinite cohomology in the present context. In section 5 we prove the main result linking that definition to the general categorical construction of section 2. In section 6 we discuss the example of a small quantum group.

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2. Morphisms through a category

2.1. **Generalities.** Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ be small categories, and $\Phi : \mathcal{B} \to \mathcal{A}$, $\Phi' : \mathcal{B} \to \mathcal{A}'$ be functors. Fix $X \in Ob(\mathcal{A})$, $Y \in Ob(\mathcal{A}')$. We define the set of "morphisms from X to Y through $\mathcal{B}$" as π_0 of the category of diagrams

$$X \longrightarrow \Phi(Z); \quad \Phi'(Z) \longrightarrow Y, \qquad Z \in \mathcal{B}. \tag{2}$$

This set will be denoted by $Hom_{\mathcal{A}_\mathcal{B}\mathcal{A}'}(X,Y)$. Thus elements of $Hom_{\mathcal{A}_\mathcal{B}\mathcal{A}'}(X,Y)$ are diagrams of the form (2), with two diagrams identified if there exists a morphism between them.

If the categories and the functors are additive (respectively, R-linear for a commutative ring R), then $Hom_{\mathcal{A}_\mathcal{B}\mathcal{A}'}(X,Y)$ is an abelian group (respectively, an R-module); to add two diagrams of the form (2) one sets $Z = Z_1 \oplus Z_2$ with the obvious arrows.

We have the composition map

$$Hom_{\mathcal{A}}(X', X) \times Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y) \times Hom_{\mathcal{A}'}(Y, Y') \to Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X', Y');$$

in particular, in the additive setting $Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y)$ is an $End(X) - End(Y)$ bimodule.

2.2. **Pro/Ind representable case.** If the left adjoint functor Φ_L to Φ is defined on X, then we have

$$Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y) = Hom_{\mathcal{A}'}(\Phi'(\Phi_L(X)), Y),$$

because in this case the above category contracts to the subcategory of diagrams of the form

$$X \xrightarrow{can} \Phi(\Phi_L(X)); \quad \Phi'(\Phi_L(X)) \to Y,$$

where *can* stands for the adjunction morphism. If the right adjoint functor Φ'_R is defined on Y, then

$$Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}}(X, Y) = Hom_{\mathcal{A}}(X, \Phi(\Phi'_R(Y)))$$

for similar reasons.

More generally, we have

Proposition 1. *Fix $X \in \mathcal{A}$ and $Y \in \mathcal{A}'$. Assume that the functor $F_X : \mathcal{B} \to \mathrm{Sets}$, $Z \mapsto Hom_{\mathcal{A}}(X, \Phi(Z))$ can be represented as a filtered inductive limit of representable functors $Z \mapsto Hom_{\mathcal{B}}(\iota(S), Z)$, where $S \in \mathcal{I}$ and $\iota : \mathcal{I} \to B$ is a functor between small categories. Then we have*

$$Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y) = \varinjlim_{S \in \mathcal{I}} Hom_{\mathcal{A}'}(\Phi'\iota(S), Y).$$

Alternatively, assume that the functor $F_Y : \mathcal{B}^{op} \to \mathrm{Sets}$, $Z \mapsto Hom(\Phi'(Z), Y)$ can be represented as a filtered inductive limit of representable functors $Z \mapsto Hom_{\mathcal{B}}(Z, \iota(S))$, where $S \in \mathcal{I}$. Then

$$Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y) = \varinjlim_{S \in \mathcal{I}} Hom_{\mathcal{A}}(X, \Phi\iota(S)).$$

$\square$

Remark 1. We will only use the Proposition in the case when the category $\mathcal{I}$ is the ordered set of positive (or negative) integers.

Remark 2. The assumptions of the Proposition can be rephrased by saying, in the first case, that the functor F_X is represented by the pro-object $\varprojlim \iota$, and in the second case, that the functor F_Y is represented by the ind-object $\varinjlim \iota$.

Remark 3. The results of the Proposition can be further generalized as follows. Fix $X \in \mathcal{A}$ and $Y \in \mathcal{A}'$; let $\iota : \mathcal{B}' \to B$ be a functor between small categories. Assume that either for any morphism $X \to \Phi(Z)$ the category of pairs of morphisms $X \to \Phi\iota(S)$, $\iota(S) \to Z$ making the triangle $X \to \Phi\iota(S) \to \Phi(Z)$ commutative is non-empty and connected, or for any morphism $\Phi'(Z) \to Y$ the category of pairs of morphisms $Z \to \iota(S)$, $\Phi'\iota(S) \to Y$ making the triangle $\Phi'(Z) \to \Phi'\iota(S) \to Y$ commutative is non-empty and connected. Then the natural map $Hom_{\mathcal{A}_{\mathcal{B}'}\mathcal{A}'}(X, Y) \to Hom_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y)$ is an isomorphism.

Example 1. Let M be a Noetherian scheme, and $\mathcal{A} = \mathcal{A}' = D^b(Coh_M)$ be the bounded derived category of coherent sheaves on M; let $\Phi = \Phi' : \mathcal{B} \hookrightarrow \mathcal{A}$ be the full embedding of the subcategory of complexes whose cohomology is supported on a closed subset $i : N \hookrightarrow M$. Then the right adjoint functor $i_* \circ i^!$ is well-defined as a functor to a "larger" derived category of quasi-coherent sheaves, while the left adjoint functor $i_* \circ i^*$ is a well-defined functor to the Grothendieck-Serre dual category, the derived category of pro-coherent sheaves (introduced in Deligne's appendix to [H]).

Let $C^\bullet$ be a complex of coherent sheaves representing the object $X \in D^b(Coh_M)$. Let X_n be the object in the derived category represented by the complex $C^i_n = C^i \otimes \mathcal{O}_M/\mathcal{J}^n_N$ (the nonderived tensor product) where $\mathcal{J}_N$ is the ideal sheaf of N. For $\mathcal{F} \in \mathcal{B}$ we have $\varinjlim Hom(X_n, \mathcal{F}) \widetilde{\longrightarrow} Hom(X, \mathcal{F})$. Thus applying Proposition 1 to $\iota : \mathbb{Z}_+ \to \mathcal{B}$ given by $\iota : n \mapsto X_n$, we get:

$$Hom_{\mathcal{A}_\mathcal{B}\mathcal{A}}(X, Y) = \varinjlim Hom(X_n, Y) = Hom(i_*(i^*(X)), Y) = Hom(X, i_*(i^!(Y))).$$

In particular, if $X = \mathcal{O}_M$ is the structure sheaf, we get

$$Hom_{\mathcal{A}_\mathcal{B}}(\mathcal{O}_M, Y[i]) = H^i_N(Y), \tag{3}$$

where $H^\bullet_N(Y)$ stands for cohomology with support on N (see e.g. [H]).

2.3. **Triangulated full embeddings.** In all examples below $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ will be triangulated, and Φ, Φ' will be full embeddings of a thick subcategory. Assume that this is the case, and moreover $\mathcal{A} = \mathcal{A}'$, $\Phi = \Phi'$.

Proposition 2. *We have a long exact sequence*

$$Hom_{\mathcal{A}_\mathcal{B}\mathcal{A}}(X, Y) \to Hom_\mathcal{A}(X, Y) \to Hom_{\mathcal{A}/\mathcal{B}}(X, Y) \to Hom_{\mathcal{A}_\mathcal{B}\mathcal{A}}(X, Y[1]).$$

Proof. The connecting homomorphism $Hom_{\mathcal{A}/\mathcal{B}}(X, Y) \to Hom_{\mathcal{A}_\mathcal{B}\mathcal{A}}(X, Y[1])$ is constructed as follows. Let $X \leftarrow X' \to Y$ be a fraction of morphisms in $\mathcal{A}$ representing a morphism $X \to Y$ in $\mathcal{A}/\mathcal{B}$; the cone K of the morphism $X' \to X$ belongs to $\mathcal{B}$. Assign to this fraction the diagram $X \to K;\ K \to Y[1]$, where the morphism $K \to Y[1]$ is defined as the composition $K \to X'[1] \to Y[1]$.

All the required verifications are straightforward; the hardest one is to check that the sequence is exact at the term $Hom_{\mathcal{A}/\mathcal{B}}(X, Y)$. Here one shows that for any two diagrams $X \to K';\ K' \to Y[1]$ and $X \to K'';\ K'' \to Y[1]$ connected by a morphism $K' \to K''$ making the two triangles commute, and for any two fractions $X \leftarrow X' \to Y$ and $X \leftarrow X'' \to Y$ to which the connecting homomorphism assigns the respective diagrams, one can construct a morphism $X' \to X''$ making the triangle formed by X', X'', X commutative, and the triangle formed by X', X'', Y will then commute up to a morphism $X \to Y$. □

3. Algebra $A^\#$ and modules over it

All algebras below will be associative and unital algebras over a field $\mathbf{k}$.

3.1. **The set-up.** *We make the following assumptions.* A $\mathbb{Z}$-graded finite dimensional algebra A and graded subalgebras $K = A^0$, $B = A^{\leq 0}$, $N = A^{\geq 0} \subset A$ are fixed and satisfy the following conditions:

(1) $B = A^{\leq 0}$, $N = A^{\geq 0}$ are graded by, respectively, $\mathbb{Z}^{\leq 0}$, $\mathbb{Z}^{\geq 0}$, and $K = B \cap N$ is the component of degree 0 in N.

(2) $K = A^0$ is semisimple and the map $N \otimes_K B \to A$ provided by the multiplication map is an isomorphism.

(3) Consider the K-N-bimodule $N^\vee = Hom_{K^{op}}(N, K)$. We require that the tensor product $S = N^\vee \otimes_N A$ is an injective right N-module.

3.2. **N-modules, $N^\vee$-comodules, and $N^\#$-modules.** By a "module" we will mean a finite dimensional graded left module, unless stated otherwise (though all the results of this section are also applicable to ungraded or infinite dimensional modules).

Since N is a finitely generated projective right K-module, the K-bimodule $N^\vee$ has a natural structure of a coring, i.e., there is a comultiplication map $N^\vee \to N^\vee \otimes_K N^\vee$ and a counit map $N^\vee \to K$ satisfying the usual coassociativity and counity conditions. Consequently, there is a natural algebra structure on $N^\# = Hom_{K^{op}}(N^\vee, K)$ and an injective morphism of algebras $K \to N^\#$. The category of right N-modules is isomorphic to the category of right $N^\vee$-comodules and the category of left $N^\#$-modules is isomorphic to the category of left $N^\vee$-comodules. In particular, $N^\vee$ is an $N^\#$-N-bimodule.

Recall that the cotensor product $P \,\square_{N^\vee} Q$ of a right $N^\vee$-comodule P and a left $N^\vee$-comodule Q is defined as the kernel of the pair of maps $P \otimes_K Q \rightrightarrows P \otimes_K N^\vee \otimes_K Q$ one of which is induced by the coaction map $P \to P \otimes_K N^\vee$ and the other by the coaction map $Q \to N^\vee \otimes_K Q$. There are natural isomorphisms $P \,\square_{N^\vee} N^\vee \cong P$ and $N^\vee \,\square_{N^\vee} Q \cong Q$.

Proposition 3. *a) i) For any right N-module P and any left N-module Q there is a natural map of $\mathbf{k}$-vector spaces $P \otimes_N Q \to P \,\square_{N^\vee} (N^\vee \otimes_N Q)$, which is an isomorphism, at least, when P is injective or Q is projective.*

ii) For any right N-module P and any left $N^\#$-module Q there is a natural map of $\mathbf{k}$-vector spaces $P \otimes_N (N \,\square_{N^\vee} Q) \to P \,\square_{N^\vee} Q$, which is an isomorphism, at least, when P is projective or Q is injective.

b) The functors $P \mapsto N^\vee \otimes_N P$ and $M \mapsto N \,\square_{N^\vee} M$ are mutually inverse equivalences between the categories of projective left N-modules and injective left $N^\#$-modules.

c) The functors $P \mapsto N^\vee \otimes_N P$ and $M \mapsto N \,\square_{N^\vee} M$ are mutually inverse tensor equivalences between the tensor category of N-bimodules that are projective left N-modules with the operation of tensor product over N and the tensor category of $N^\#$-N-bimodules that are injective left $N^\#$-modules with the operation of cotensor product over $N^\vee$.

Proof. Both assertions of (a) state existence of associativity (iso)morphisms connecting the tensor and cotensor products. In particular, in (i) we have to construct a natural map $(P \,\square_{N^\vee} N^\vee) \otimes_N Q \to P \,\square_{N^\vee} (N^\vee \otimes_N Q)$. More generally, let us consider an arbitrary $N^\#$-N-bimodule R and construct a natural map $(P \,\square_{N^\vee} R) \otimes_N Q \to P \,\square_{N^\vee} (R \otimes_N Q)$. This map can be defined in two equivalent ways. The first approach is to take the tensor product of the exact sequence of right N-modules $0 \to P \,\square_{N^\vee} R \to P \otimes_K R \to P \otimes_K N^\vee \otimes_K R$ with the left N-module Q. Since the resulting sequence is a complex, there exists a unique map $(P \,\square_{N^\vee} R) \otimes_N Q \to P \,\square_{N^\vee} (R \otimes_N Q)$ making a commutative triangle with the natural maps of $(P \,\square_{N^\vee} R) \otimes_N Q$ and $P \,\square_{N^\vee} (R \otimes_N Q)$ into $P \otimes_K R \otimes_N Q$. It is clear that this map is an isomorphism whenever Q is a flat N-module. Analogously, for any P, Q, R there is a natural isomorphism $(P \,\square_{N^\vee} R) \otimes_K Q \cong P \,\square_{N^\vee} (R \otimes_K Q)$, since K is semisimple. The second way is to take the cotensor product of the exact sequence of left $N^\vee$-comodules $R \otimes_K N \otimes_K Q \to R \otimes_K Q \to R \otimes_N Q \to 0$ with the

right $N^\vee$-comodule P. Again, since the resulting sequence is a complex, there exists a unique map $(P\,\square_{N^\vee} R)\otimes_N Q \to P\,\square_{N^\vee}(R\otimes_N Q)$ making a commutative triangle with the natural maps from $P\,\square_{N^\vee} R\otimes_K Q$ to $(P\,\square_{N^\vee} R)\otimes_N Q$ and $P\,\square_{N^\vee}(R\otimes_N Q)$. Clearly, this map is an isomorphism whenever P is a coflat $N^\vee$-comodule (i.e., the cotensor product with P preserves exactness). Now any injective right N-module is a coflat right $N^\vee$-comodule, since it is a direct summand of a direct sum of copies of $N^\vee$. The two associativity maps that we have constructed coincide, since the relevant square diagram commutes. The proof of (ii) is analogous.

To prove (b), notice the isomorphisms $N\,\square_{N^\vee}(N^\vee\otimes_N P)\cong N\otimes_N P\cong P$ and $N^\vee\otimes_N(N\,\square_{N^\vee} M)\cong N^\vee\,\square_{N^\vee} M\cong M$ for a projective left N-module P and an injective left $N^\#$-module M. Since a projective left N-module is a direct summand of an N-module of the form $N\otimes_K V$ and an injective left $N^\#$-module is a direct summand of an $N^\#$-module of the form $\mathrm{Hom}_K(N^\#, V)\cong N^\vee\otimes_K V$ for a K-module V, the functors in question transform projective N-modules to injective $N^\#$-modules and vice versa.

To deduce (c), notice the isomorphism $(N^\vee\otimes_N P)\,\square_{N^\vee}(N^\vee\otimes_N Q)\cong N^\vee\otimes_N P\otimes_N Q$ for a N-bimodule P and a projective left N-module Q. It is straightforward to check that this isomorphism preserves the associativity constraints. $\square$

3.3. **Definition of $A^\#$.** It follows from the condition (2) that A is a projective left N-module. By Proposition 3(c), the tensor product $S = N^\vee\otimes_N A$ is a ring object in the tensor category of $N^\vee$-bicomodules with respect to the cotensor product over $N^\vee$. By the condition (3) and the right analogue of Proposition 3(c), the cotensor product $A^\# = S\,\square_{N^\vee} N^\#$ is a ring object in the tensor category of $N^\#$-bimodules with respect to the tensor product over $N^\#$. The embedding $N\to A$ induces injective maps $N^\vee\to S$ and $N^\#\to A^\#$; these are unit morphisms of the ring objects in the corresponding tensor categories. So $A^\#$ has a natural associative algebra structure and $N^\#$ is identified with a subalgebra in $A^\#$. Notice that $A^\#$ is a projective right $N^\#$-module by the definition.

Proposition 4. *There is a natural isomorphism between the $N^\#$-A-bimodule $S = N^\vee\otimes_N A$ and the $A^\#$-N-bimodule $S^\# = A^\#\otimes_{N^\#} N^\vee$, making S an $A^\#$-A-bimodule. Moreover, there are isomorphisms:*

$$A^\#\cong End_{A^{op}}(S),\quad A^{op}\cong End_{A^\#}(S^\#).$$

Proof. By the definition, we have $S^\# = (S\,\square_{N^\vee} N^\#)\otimes_{N^\#} N^\vee\cong S\,\square_{N^\vee} N^\vee\cong S$, since S is an injective right N-module. Let us show that the right A-module and the left $A^\#$-module structures on $S\cong S^\#$ commute. The isomorphism $S\otimes_N A\cong S\otimes_N(N\,\square_{N^\vee} S)\cong S\,\square_{N^\vee} S$ transforms the right action map $S\otimes_N A\to S$ into the map $S\,\square_{N^\vee} S\to S$ defining the structure of ring object in the tensor category of $N^\vee$-bicomodules on S. Analogously, the isomorphisms $A^\#\otimes_{N^\#} S^\#\cong(S\,\square_{N^\vee} N^\#)\otimes_{N^\#} S\cong S\,\square_{N^\vee} S$ and $S^\#\cong S$ transform the left action map $A^\#\otimes_{N^\#} S^\#\to S^\#$ into the same map $S\,\square_{N^\vee} S\to S$. Finally, there is an isomorphism $A^\#\otimes_{N^\#} S\otimes_N A\cong(S\,\square_{N^\vee} N^\#)\otimes_{N^\#} S\otimes_N(N\,\square_{N^\vee} S)\cong S\,\square_{N^\vee} S\,\square_{N^\vee} S$, so the right and left actions commute since S is an associative ring object in the tensor category of $N^\vee$-bicomodules. Now we have $Hom_{A^{op}}(N^\vee\otimes_N A, N^\vee\otimes_N A)\cong Hom_{N^{op}}(N^\vee, N^\vee\otimes_N A)\cong(N^\vee\otimes_N A)\,\square_{N^\vee} N^\# = A^\#$ and $Hom_{A^\#}(A^\#\otimes_{N^\#} N^\vee, A^\#\otimes_{N^\#} N^\vee)\cong Hom_{N^\#}(N^\vee, A^\#\otimes_{N^\#} N^\vee)\cong N\,\square_{N^\vee}(A^\#\otimes_{N^\#} N^\vee)\cong A$. $\square$

3.4. **N-projective (injective) modules.** By $A-mod$ we denote the category of (graded finite dimensional) left A-modules.

Consider the full subcategories $A-mod_{N-proj} \subset A-mod$, $A^{\#}-mod_{N^{\#}-inj} \subset A^{\#}-mod$ consisting of modules whose restriction to N is projective (respectively, restriction to $N^{\#}$ is injective).

We abbreviate $D(A) = D^b(A-mod)$, $D(A^{\#}) = D^b(A^{\#}-mod)$, and let $D_{\infty/2}(A) \subset D(A)$, $D_{\infty/2}(A^{\#}) \subset D(A^{\#})$ be the full triangulated subcategories generated by $A-mod_{N-proj}$, $A^{\#}-mod_{N^{\#}-inj}$ respectively.

Proposition 5. *We have canonical equivalences:* $A-mod_{N-proj} \cong A^{\#}-mod_{N^{\#}-inj}$, $D_{\infty/2}(A) \cong D_{\infty/2}(A^{\#})$.

Proof. Let us show that the adjoint functors $P \mapsto S\otimes_A P$ and $M \mapsto Hom_{A^{\#}}(S,M)$ between the categories $A-mod$ and $A^{\#}-mod$ induce an equivalence between their full subcategories $A-mod_{N-proj}$ and $A^{\#}-mod_{N^{\#}-inj}$. It suffices to check that the adjunction morphisms $P \to Hom_{A^{\#}}(S,\ S\otimes_A P)$ and $S\otimes_A Hom_{A^{\#}}(S,M) \to M$ are isomorphisms when an A-module P is projective over N and an $A^{\#}$-module M is injective over $N^{\#}$. There are natural isomorphisms $S\otimes_A P \cong N^\vee \otimes_N P$ and $Hom_{A^{\#}}(S,M) \cong Hom_{N^{\#}}(N^\vee, M) \cong N \,\square_{N^\vee} M$, so it remains to apply Proposition 3(b). To obtain the equivalence of categories $D_{\infty/2}(A) \cong D_{\infty/2}(A^{\#})$, it suffices to check that $D_{\infty/2}(A)$ is equivalent to the bounded derived category of the exact category $A-mod_{N-proj}$ and $D_{\infty/2}(A^{\#})$ is equivalent to the bounded derived category of the exact category $A^{\#}-mod_{N^{\#}-inj}$. Let us prove the former; the proof of the latter is analogous. It suffices to check that for any bounded complex of N-projective A-modules P and any bounded complex of A-modules X together with a quasi-isomorfism $X \to P$ there exists a bounded complex of N-projective A-modules Q together with a quasi-isomorphism $Q \to X$. Let Q' be a bounded above complex of projective A-modules mapping quasi-isomorphically into X; then the canonical truncation $Q'_{\geq -n}$ for large enough n provides the desired complex Q. $\square$

3.5. **The case of an invertible entwining map.** Consider the multiplication map $\phi : B\otimes_K N \to A \cong N\otimes_K B$. It yields a map $\psi : N^\vee \otimes_K B \to Hom_{K^{op}}(N,B) \cong B\otimes_K N^\vee$. Assume that the map ψ is an isomorphism and consider the inverse map $\psi^{-1} : B\otimes_K N^\vee \to N^\vee \otimes_K B$. By the analogous "lowering of indices" we obtain from it a map $N^{\#}\otimes_K B \to Hom_{K^{op}}(N^\vee, B) = B\otimes_K N^{\#}$ that will be denoted by $\phi^{\#}$.

Then the algebra $A^{\#}$ can be also defined as the unique associative algebra with fixed embeddings of $N^{\#}$ and B into $A^{\#}$ such that

i) the embeddings $N^{\#} \to A^{\#}$ and $B \to A^{\#}$ form a commutative square with the embeddings $K \to N^{\#}$ and $K \to B$;

ii) the multiplication map induces an isomorphism $B\otimes_K N^{\#} \to A^{\#}$;

iii) the map induced by the multiplication map $N^{\#}\otimes_K B \to A^{\#} \cong B\otimes_K N^{\#}$ coincides with $\phi^{\#}$.

Indeed, the existence of an algebra A with subalgebras N and B in terms of which the map ϕ is defined can be easily seen to be equivalent to the map ψ satisfying the equations of a right entwining structure for the coring $N^\vee$ and the algebra B (see [BW] or [P] for the definition). When ψ is invertible, it is a right entwining structure if and only if ψ^{-1} is a left entwining structure, and the latter is equivalent to the existence of an algebra $A^{\#}$ satisfying (i-iii).

To show that the two definitions of $A^\#$ are equivalent, it suffices to check that the ring object S in the tensor category of $N^\vee$-bicomodules can be constructed in terms of the entwining structure ψ in the way explained in [Brz] or [P].

3.6. **The case of a self-injective N.** Assume that N is self-injective. In this case $A^\#$ is canonically Morita equivalent to A; the equivalence is defined by the $A^\#$-A-bimodule S, so it sends $A - mod_{N-proj} = A - mod_{N-inj}$ to $A^\# - mod_{N^\#-proj} = A^\# - mod_{N^\#-inj}$.

Indeed, $N^\vee$ is obviously an injective generator of the category of right N-modules. Since every injective N-module is projective, $N^\vee$ is a projective right N-module. Since N is an injective right N-module, it is a direct summand of a finite direct sum of copies of $N^\vee$. So $N^\vee$ is a projective generator of the category of right N-modules; hence $S = N^\vee \otimes_N A$ is a projective generator of the category of right A-modules. Now it remains to use Proposition 4. Analogously, $N^\#$ is Morita equivalent to N; hence $N^\#$ is also self-injective.

If N is Frobenius, $N^\#$ is isomorphic to N and $A^\#$ is isomorphic to A. Indeed, K is also Frobenius; choose a Frobenius linear function $K \to \mathbf{k}$; then the right K-module $\mathrm{Hom}_{\mathbf{k}}(K, \mathbf{k})$ is isomorphic to K. Hence a Frobenius linear function $N \to \mathbf{k}$ lifts to a right K-module map $N \to K$. Now the composition $N \otimes_K N \to N \to K$ of the multiplication map $N \otimes_K N \to N$ and the right K-module map $N \to K$ defines an isomorphism of right N-modules $N \to Hom_{K^{op}}(N, K) = N^\vee$. By Proposition 4, this leads to the isomorphism $A^\# \cong A$ and analogously to the isomorphism $N^\# \cong N$; these isomorphisms are compatible with the embeddings $N \to A$ and $N^\# \to A^\#$, but not with the embeddings of K to N and $N^\#$, in general.

4. Definitions of $Ext^{\infty/2}$ by explicit complexes

4.1. **Concave and convex resolutions.** A complex of graded modules will be called *convex* if the grading "goes down", i.e. for any $n \in \mathbb{Z}$ the sum of graded components of degree more than n is finite dimensional; it will be called *non-strictly convex* if the grading "does not go up", i.e. the graded components of high enough degree vanish. A complex of graded modules will be called *concave* (respectively *non-strictly concave*) if the grading "goes up" (respectively "does not go down") in the similar sense.

An $A^\#$-module M will be called *weakly projective relative to* $N^\#$ if for any $A^\#$-module J which is injective as an $N^\#$-module one has $Ext^i_{A^\#}(M, J) = 0$ for all $i \neq 0$. Analogously one defines A-modules *weakly injective relative to* N. Notice that any $A^\#$-module induced from an $N^\#$-module is weakly projective relative to $N^\#$. The class of $A^\#$-modules weakly projective relative to $N^\#$ is closed under extensions and kernels of surjective morphisms.

Lemma 1. *i) Any A-module admits a left concave resolution by A-modules which are projective as N-modules. Any $A^\#$-module admits a left non-strictly convex resolution by $A^\#$-modules which are weakly projective relative to $N^\#$.*

ii) Any finite complex of A-modules is a quasiisomorphic quotient of a bounded above concave complex of N-projective A-modules. Any finite complex of $A^\#$-modules is a quasiisomorphic quotient of a bounded above non-strictly convex complex of $A^\#$-modules weakly projective relative to $N^\#$.

Proof. To deduce (ii) from (i) choose a quasiisomorphic surjection onto a given complex $C^\bullet \in Com^b(A{-}mod)$ from a complex of A-projective modules $P^\bullet \in Com^-(A{-}mod)$ (notice that condition (2) of 3.1 implies that an A-projective module is also N-projective), and apply (i) to the module of cocycles $Z^n = P^{-n}/d(P^{-n-1})$ for large n.

To check (i) it suffices to find for any $M \in A-mod$ a surjection $P \twoheadrightarrow M$, where P is N-projective, and if n is such that all graded components M_i for $i < n$ vanish, then $P_i = 0$ for $i < n$ and $P_n \widetilde{\longrightarrow} M_n$. It suffices to take $P = Ind_B^A(Res_B^A(M))$. It is indeed N-projective, because of the equality

$$Res_N^A(Ind_B^A(M)) = Ind_K^N(M)), \tag{4}$$

which is a consequence of assumption (2). □

The second assertions of (i) and (ii) are proven in the analogous way, except that one uses the induction from $N^\#$ (this is even simpler, as weak relative projectivity of the relevant modules is just obvious).

4.2. **Definition of semi-infinite Ext's.**

Definition 1. (cf. [FS], §2.4) The assumptions (1–3) of 3.1 are enforced. Let $X \in D(A^\#)$ and $Y \in D(A)$. Let $P_\swarrow^X$ be a non-strictly convex bounded above complex of $A^\#$-modules weakly projective relative to $N^\#$ that is quasiisomorphic to X, and $P_\nwarrow^Y$ be a concave bounded above complex of N-injective A-modules that is quasiisomorphic to Y. Then we set

$$Ext^{\infty/2+i}(X,Y) = H^i(Hom^\bullet_{A^\#}(P_\swarrow^X, S \otimes_A P_\nwarrow^Y)). \tag{5}$$

Independence of the right-hand side of (5) on the choice of $P_\swarrow^X$, $P_\nwarrow^Y$ follows from Theorem 1 below.

Remark 4. Notice that Hom in the right-hand side of (5) is Hom in the category of graded modules. As usual, it is often convenient to denote by $Ext^{\infty/2+i}(X,Y)$ the graded space which in present notations is written down as $\bigoplus\limits_n Ext^{\infty/2+i}(X, Y(n))$, where (n) refers to the shift of grading by $-n$.

Remark 5. Definition 1 is compatible with [Ar1], Definition 3.3.6 in the sense explained below. In this remark we will freely use the notation of *loc. cit.*

For a finite dimensional algebra A the definition of the algebra $A^\#$ given in [Ar1], 3.3.2 reduces to $A^\# = End_{A^{op}}(S)$, where S is defined by $S = Hom_k(N,k) \otimes_N A$, so according to Proposition 4 this agrees with our definition (see also 3.5). Notice that in *loc. cit.* it is presumed that $K = k$, so one has $N^\# = N$.

Let $L \in Com^b(A^\# - mod)$ and $M \in Com^b(A - mod)$. Then the restricted Bar-resolution $\mathrm{Bar}^\bullet(A^\#, N^\#, L)$ is a non-strictly convex bounded above resolution of L by $A^\#$-modules weakly projective relative to $N^\#$; and $\mathrm{Bar}^\bullet(A,B,M)$ is a concave bounded above resolution of M by N-projective A-modules. Thus the definition of semi-infinite cohomology

$$Ext^{\infty/2+i}(L,M) = Hom^\bullet_{A^\#}\left(\mathrm{Bar}^\bullet(A^\#, N^\#, L), S \otimes_A \mathrm{Bar}^\bullet(A,B,M)\right)$$

from *loc. cit.* is a particular case of our definition whenever both are applicable.

4.3. **Alternative assumptions.** The conditions on the resolutions $P^X_\swarrow$, $P^Y_\nwarrow$ used in (5) are formulated in terms of the subalgebras $N \subset A$ and $N^\# \subset A^\#$; the subalgebra $B \subset A$ is not mentioned there (and the left-hand side of (9) in Theorem 1 below does not depend on it either). However, existence of a "complemental" subalgebra B is used in the construction of a resolution $P^Y_\nwarrow$ with required properties. Moreover, the next standard Lemma shows that conditions on the resolutions $P^X_\swarrow$, $P^Y_\nwarrow$ can be alternatively rephrased in terms of the subalgebra B and any nonpositively graded subalgebra $B^\# \subset A^\#$ such that $B^\# \otimes_K N^\# \cong A^\#$, assuming that such a subalgebra exists (e.g., in the assumptions of 3.5 or when N is Frobenius and $B \otimes_K N \cong A$).

Lemma 2. *i) An A-module is N-projective iff it has a filtration with subquotients of the form $Ind_B^A(M)$, $M \in B - mod$.*

ii) Assume that $B^\# \subset A^\#$ is a subalgebra graded by nonpositive integers such that $K \subset B^\#$ and the multiplication map induces an isomorphism $B^\# \otimes_K N^\# \to A^\#$. Then an $A^\#$-module is $N^\#$-injective iff it has a filtration with subquotients of the form $CoInd_{B^\#}^{A^\#}(M)$, $M \in B^\# - mod$.

Proof. The "if" direction follows from semisimplicity of K, and equality (4) above. To show the "only if" part let M be a projective N-module. Let M^- be its graded component of minimal degree; then the canonical morphism

$$Ind_K^N M^- \to M \tag{6}$$

is injective. If M is actually an A-module, then the injection $M^- \to M$ is an embedding of B-modules, hence yields a morphism of A-modules

$$Ind_B^A M^- \to M. \tag{7}$$

(4) shows that Res_N^A sends (7) into (6); in particular (7) is injective. Thus the bottom submodule of the required filtration is constructed, and the proof is finished by induction. The proof of (ii) is analogous. □

Remark 6. Replacing the assumption of existence of a subalgebra $B \subset A$ (assuming only that A is a projective left N-module) with the assumption of existence of a nonpositively graded subalgebra $B^\# \subset A^\#$ such that $B^\# \otimes_K N^\# \cong A^\#$, one can define $Ext^{\infty/2+i}(X, Y)$ in terms of injective resolutions rather than projective ones. Namely, for $X \in D(A^\#)$ and $Y \in D(A)$, let $J^X_\searrow$ be a convex bounded below complex of $N^\#$-injective modules quasiisomorphic to X, and $J^Y_\nearrow$ be a non-strictly concave bounded below complex of A-modules weakly injective relative to N. Then set

$$Ext^{\infty/2+i}(X, Y) = H^i(Hom^\bullet(Hom_{A^\#}(S, J^X_\searrow), J^Y_\nearrow)).$$

The analogue of Theorem 1 below holds for this definition as well, hence it follows that the two definitions are equivalent whenever both are applicable.

4.4. **Comparison with ordinary Ext and Tor.** In four special cases $Ext^{\infty/2+i}(X, Y)$ coincides with a combination of traditional derived functors. First, suppose that $Res_N^A(Y)$ has finite projective dimension; then one can use a finite complex $P^Y_\nwarrow$ in (5) above. It follows immediately, that in this case we have

$$Ext^{\infty/2+i}(X, Y) \cong Hom_{D(A^\#)}(X, S \overset{L}{\otimes}_A Y[i]).$$

Analogously, in the assumptions of Remark 6 above, whenever $Res^{A^\#}_{N^\#}(X)$ has finite injective dimension one has

$$Ext^{\infty/2+i}(X,Y) \cong Hom_{D(A)}(RHom_{A^\#}(S,X), Y[i]).$$

On the other hand, suppose that the complex $P^X_{\swarrow}$ in (5) can be chosen to be a finite complex of $A^\#$-modules whose terms have filtrations with subquotients being $A^\#$-modules induced from $N^\#$-modules. We claim that in this case we have

$$Ext^{\infty/2+i}(X,Y) \cong H^i(RHom_{A^\#}(X,S) \overset{L}{\otimes}_A Y).$$

This isomorphism is an immediate consequence of the next Lemma. Analogously, in the situation of Remark 6, whenever $J^Y_{\nearrow}$ can be chosen to be a finite complex of A-modules whose terms have filtrations with subquotients being A-modules coinduced from N-modules, one has

$$Ext^{\infty/2+i}(X,Y) \cong H^i(X^* \overset{L}{\otimes}_{A^\#} RHom_A(S^*,Y)).$$

Here we denote by $V \mapsto V^*$ the passage to the dual vector space, $V^* = Hom_k(V,k)$, and the corresponding functor on the level of derived categories.

Lemma 3. *Let $L \in A^\# - mod$, $M \in A - mod$ be such that L has a filtration with subquotients being $A^\#$-modules induced from $N^\#$-modules, while M is N-projective. Then we have*

a) i) $Ext^i_{A^\#}(L,S) = 0$ and $Tor^A_i(Hom_{A^\#}(L,S), M) = 0$ for $i \neq 0$.

ii) $Tor^A_i(S,M) = 0$ and $Ext^i_{A^\#}(L, S \otimes_A M) = 0$ for $i \neq 0$.

b) The natural map

$$Hom_{A^\#}(L,S) \otimes_A M \longrightarrow Hom_{A^\#}(L, S \otimes_A M) \tag{8}$$

is an isomorphism.

Proof. The first equality in (i) holds because S is an injective $N^\#$-module. To check the second one, notice that if $L = Ind^A_{N^\#} L_0$, then $Hom_{A^\#}(L,S) \cong Hom_{N^\#}(L_0, N^\vee \otimes_N A) \cong Hom_{N^\#}(L_0, N^\vee) \otimes_N A$ is a right A-module induced from a right N-module. The first equality in (ii) holds because the right A-module S is induced from a right N-module, and the second one is verified since $S \otimes_A M$ is $N^\#$-injective. Let us now deduce (b) from (a). Notice that (a) implies that both sides of (8) are exact in M (and also in L), i.e. send exact sequences $0 \to M' \to M \to M'' \to 0$ with M', M'' being N-projective into exact sequences. Also (8) is evidently an isomorphism for $M = A$. For any N-projective M there exists an exact sequence

$$A^n \xrightarrow{\phi} A^m \to M \to 0$$

with the image and kernel of ϕ being N-projective. Thus both sides of (8) turn into exact sequences, which shows that (8) is an isomorphism for any N-projective M. □

5. Main result

Theorem 1. *Let $D_{\infty/2} \subset D(A^\#)$, $D_{\infty/2} \subset D(A)$ be the full triangulated subcategory of $D(A^\#)$ generated by $N^\#$-injective modules, which is equivalent to the*

full triangulated subcategory of $D(A)$ generated by N-projective modules. For $X \in D^b(A^\# - mod)$, $Y \in D^b(A - mod)$ we have a natural isomorphism

$$Hom_{D(A^\#)_{D_{\infty/2}}D(A)}(X, Y[i]) \cong Ext^{\infty/2+i}(X, Y). \tag{9}$$

Example 2. [1] Assume that $A = N$ is a Frobenius algebra and $K = \mathbf{k}$. Then $A^\# \cong A$, and according to section 4.4 we have $Ext^{\infty/2+i}(X, Y) = Tor^A_{-i}(X^*, Y)$. In this case we can identify $\mathcal{A}'$ with $\mathcal{A}$, so that $\Phi' = \Phi$ is the embedding of the category of perfect complexes. The long exact sequence of Proposition 2 becomes a standard sequence linking Ext, Tor and Hom in the stable category $A/\mathcal{B}$; in particular, for modules over a finite group we recover the description of Tate cohomology as Hom's in the stable category.

Remark 7. Notice that the definition of the left hand side in (9) applies also to non-graded algebras and modules. Thus the Theorem allows one to extend the definition of semi-infinite cohomology to nongraded algebras. Another definition of the semi-infinite cohomology of nongraded algebras was given in [P].

Let us point out that these two definitions are *not* equivalent: for example, when $\mathbf{k}$ is a finite or a countable field, the left hand side of (9) in the nongraded case is no more than countable, while the semi-infinite cohomology defined in *loc. cit.* can have the cardinality of continuum.

The proof of Theorem 1 is based on the following

Lemma 4. *i) Every N-projective A-module admits a non-strictly convex left resolution consisting of A-projective modules.*

ii) A finite complex of N-projective A-modules is quasiisomorphic to a non-strictly convex bounded above complex of A-projective modules.

Proof. (ii) follows from (i) as in the proof of Lemma 1. (Recall that, according to a well-known argument due to Hilbert, if a bounded above complex of projectives represents an object of the derived category which has finite projective dimension, then for large negative n the module of cocycles is projective.)

To prove (i) it is enough for any N-projective module M to find a surjection $Q \twoheadrightarrow M$, where Q is A-projective, and $Q_n = 0$ for $i > n$ provided $M_i = 0$ for $i > n$. (Notice that the kernel of such a surjection is N-projective, because Q is N-projective by condition (2).) We can take Q to be $Ind^A_N(Res^A_N(M))$, and the condition on grading is clearly satisfied. □

Proposition 6. *a) Let $P_{\nwarrow}$ be a concave bounded above complex of A-modules representing an object $Y \in D^-(A - mod)$. Let $P^n_{\nwarrow}$ be the $(-n)$-th stupid truncation of $P_{\nwarrow}$ (thus $P^n_{\nwarrow}$ is a subcomplex of $P_{\nwarrow}$).*

Let Z be a finite complex of N-projective A-modules. Then we have

$$Hom_{D^-(A-mod)}(Z, Y) \xrightarrow{\sim} \varinjlim Hom_{D(A)}(Z, P^n_{\nwarrow}). \tag{10}$$

In fact, for n large enough we have

$$Hom_{D^-(A-mod)}(Z, Y) \xrightarrow{\sim} Hom_{D(A)}(Z, P^n_{\nwarrow}).$$

[1]We thank A. Beilinson who suggested to us this example.

Proof. Let $Q_\swarrow$ be a non-strictly convex bounded above complex of A-projective modules quasiisomorphic to Z (which exists by Lemma 4(ii)). Then the left-hand side of (10) equals $Hom_{Hot}(Q_\swarrow, P_\nwarrow)$, where Hot stands for the homotopy category of complexes of A-modules. The conditions on gradings of our complexes ensure that there are only finitely many degrees for which the corresponding graded components both in $Q_\swarrow$ and $P_\nwarrow$ are nonzero; thus any morphism between the graded vector spaces $Q_\swarrow$, $P_\nwarrow$ factors through the finite dimensional sum of the corresponding graded components. In particular, $Hom^\bullet(Q_\swarrow, P^n_\nwarrow) \widetilde{\longrightarrow} Hom^\bullet(Q_\swarrow, P_\nwarrow)$ for large n, and hence

$$Hom_{D(A)}(Z, P^n_\nwarrow) = Hom_{Hot}(Q_\swarrow, P^n_\nwarrow) \widetilde{\longrightarrow} Hom_{Hot}(Q_\swarrow, P_\nwarrow)$$

for large n. □

Corollary 1. *Let $P_\nwarrow$ be a concave bounded above complex of N-projective A-modules, and X be the corresponding object of $D^-(A-mod)$. Then the functor on $D_{\infty/2}$ given by $Z \mapsto Hom_{D^-(A-mod)}(Z, Y)$ is represented by the ind-object $\varinjlim P^n_\nwarrow$.* □

Proof of the Theorem. We keep the notation of Definition 1. It follows from the Proposition that

$$Hom_{D(A^\#)_{D_{\infty/2}}D(A)}(X, Y[i]) = \varinjlim_n Hom_{D(A^\#)}(X, S \otimes_A (P^Y_\nwarrow)^n).$$

The right-hand side of (9) (defined in (5)) equals $H^i(Hom^\bullet_{A^\#}(P^X_\swarrow, S \otimes_A P^Y_\nwarrow))$. The conditions on gradings of $P^X_\swarrow$, $P^Y_\nwarrow$ show that for large n we have

$$Hom^\bullet_{A^\#}(P^X_\swarrow, S \otimes_A (P^Y_\nwarrow)^n) \widetilde{\longrightarrow} Hom^\bullet_{A^\#}(P^X_\swarrow, S \otimes_A P^Y_\nwarrow).$$

Since $Ext^i_{A^\#}(L, S \otimes_A M) = 0$ for $i > 0$ if L is weakly projective relative to $N^\#$ and M is N-projective, we have

$$Hom_{D(A^\#)}(X, S \otimes_A (P^Y_\nwarrow)^n) = Hom^\bullet_{A^\#}(P^X_\swarrow, S \otimes_A (P^Y_\nwarrow)^n).$$

The Theorem is proved. □

Remark 8. There is a version of Theorem 1 applicable in the situation when the condition that K is the component of degree 0 of N in (1) of 3.1 is replaced with the condition that K is the component of degree 0 of B. One just has to change the conditions on the complexes $P^X_\swarrow$, $P^Y_\nwarrow$ in Definition 1, requiring that $P^X_\swarrow$ be convex and $P^Y_\nwarrow$ be non-strictly concave, and make the related changes in the proof.

6. Semi-infinite cohomology of the small quantum group

This section concerns with the example provided by a small quantum group. So let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^-$. Let $q \in \mathbb{C}$ be a root of unity of order l, and let $A = u_q = u_q(\mathfrak{g})$ be the corresponding small quantum group [L]. We assume that l is large enough (larger than Coxeter number) and is prime to twice the maximal multiplicity of an edge in the Dynkin diagram of $\mathfrak{g}$.

Let $A^{\geq 0} = u_q^+ \subset u_q$ and $A^{\leq 0} = u_q^- \subset u_q$ be respectively the upper and the lower triangular subalgebras. The algebra u_q carries a canonical grading by the weight lattice. We fix an arbitrary element in the dual coweight lattice, which is a dominant coweight, thus we obtain a $\mathbb{Z}$-grading on u_q. Then the above conditions (1–3) are satisfied.

For an augmented $\mathbf{k}$ algebra R we write $H^\bullet(R)$ for $Ext_R(\mathbf{k},\mathbf{k})$; we abbreviate $H^\bullet = H^\bullet(u_q)$.

The cohomology algebra $H^\bullet$, and the semi-infinite cohomology $Ext^{\infty/2+\bullet}(\mathbf{k},\mathbf{k})$ were computed respectively in [GK] and [Ar1]. Let us recall the results of these computations. Below by "Hom" we will mean graded Hom as in Remark 4 above.

Let $\mathcal{N}\subset\mathfrak{g}$ be the cone of nilpotent elements, and $\mathfrak{n}\subset\mathcal{N}$ be a maximal nilpotent subalgebra. Then the Theorem of Ginzburg and Kumar asserts the existence of canonical isomorphisms

$$H^\bullet \cong \mathcal{O}(\mathcal{N}), \tag{11}$$

$$H^\bullet(u_q^+) = \mathcal{O}(\mathfrak{n}), \tag{12}$$

such that the restriction map $\mathcal{O}(\mathcal{N})\to\mathcal{O}(\mathfrak{n})$ coincides with the map arising from functoriality of cohomology with respect to maps of augmented algebras.

Also, a Theorem of Arkhipov (conjectured by Feigin) asserts that

$$Ext^{\infty/2+\bullet}(\mathbb{I},\mathbb{I}) \cong H^d_{\mathfrak{n}^-}(\mathcal{N},\mathcal{O}), \tag{13}$$

where d is the dimension of $\mathfrak{n}^-$, and $H_{\mathfrak{n}^-}$ denotes cohomology with support in $\mathfrak{n}^-$; one also has $H^i_{\mathfrak{n}^-}(\mathcal{N},\mathcal{O})=0$ for $i\neq d$.

The aim of this section is to show how (a generalization of) this isomorphism follows from Theorem 1.

6.1. $D_{\infty/2}$ **and cohomological support.** Let $D^\bullet$ denote the category defined by $Ob(D^\bullet)=Ob(D)$, $Hom_{D^\bullet}(X,Y)=Hom^\bullet(X,Y)=\bigoplus\limits_i Hom(X,Y[i])$. Then $D^\bullet$ is an $HH^\bullet$-linear category, i.e. we have a canonical homomorphism $HH^\bullet\to End(Id_{D^\bullet})$, where $HH^\bullet$ denotes the Hochschield cohomology of u_q. Since u_q is a Hopf algebra, we have a canonical homomorphism $H^\bullet\to HH^\bullet$, thus $D^\bullet$ is an $H^\bullet$ linear category. For an object $X\in D^\bullet$ its *cohomological support* $supp(X)\subset Spec(H^\bullet)$ is the set-theoretic support of the $H^\bullet$ module $End^\bullet(X)$.

Proposition 7. *For an object $X\in D$ we have:*

$$X\in D_{\infty/2} \iff supp(X)\subseteq\mathfrak{n}.$$

Proof. It is well known that $supp(X)\supset supp(Y)\cup supp(Z)$ provided that there exists a distinguished triangle $Y\to X\to Z\to Y[1]$, thus the set of objects with cohomological support contained in $\mathfrak{n}$ forms a full triangulated subcategory. In view of Lemma 2, to check the implication $\Rightarrow$ it sufficies to check that $supp(X)\subset\mathfrak{n}$ if $X=CoInd_{u_q^+}^{u_q}(M)$ for some M. For such X we have $Ext^\bullet_{u_q}(X,X)=Ext^\bullet_{u_q^+}(X,M)$. Moreover, it is not hard to check that this isomorphism is compatible with the $H^\bullet$ action, where the action on the right hand side is obtained as the composition $H^\bullet\to H^\bullet(u_q^+)\to Ext^\bullet_{u_q^+}(M,M)$ and the canonical right action of $Ext^\bullet_{u_q^+}(M,M)$. Thus in this case $Ext^\bullet_{u_q}(X,X)$ is set-theoretically supported on $\mathfrak{n}$.

Assume now that $X\in D$ is such that $supp(X)\subseteq\mathfrak{n}$. To check that $X\in D_{\infty/2}$ it suffices to show that $Ext^\bullet_{u_q^-}(M,X)$ is finite dimensional for any $M\in D^b(u_q^+-mod)$. It is a standard fact that $Ext^\bullet(M_1,M_2)$ is a finitely generated $H^\bullet(u_q^-)$ module for any $M_1,M_2\in D^b(u_q^- - mod)$, thus it suffices to see that the $H^\bullet(u_q^-)$-module $Ext^\bullet_{u_q^-}(M,X)$ is supported at $\{0\}\subset\mathfrak{n}^-=Spec(H^\bullet(u_q^-))$. This is clear, since viewed as a $H^\bullet(X)$ it is supported on $\mathfrak{n}$. $\square$

6.2. **A description of the derived u_q-modules category via coherent sheaves.** Let $\tilde{\mathcal{N}} = T^*(\mathcal{B}) = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B}, x \in rad(\mathfrak{b})\}$, where $\mathcal{B} = G/B$ is the flag variety of G identified with the set of Borel subalgebras in $\mathfrak{g}$, and rad stands for the nil-radical. Let $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ be the Springer map, $\pi : (\mathfrak{b}, x) \mapsto x$.

The result of [BL] (based on [ABG]) yields a triangulated functor $\Psi : D^b(Coh^{\mathbb{G}_m}(\tilde{\mathcal{N}})) \to D^b(u_q - Mod)$, where $\mathbb{G}_m$ acts on $\tilde{\mathcal{N}}$ by $t : (\mathfrak{b}, x) \mapsto (\mathfrak{b}, t^2 x)$, and $u_q - Mod$ stands for the category of finite dimensional modules. Notice that in contrast with the definition of $u_q - mod$, the modules in $u_q - Mod$ *do not* carry a grading.[2]

The functor satisfies the following properties:

$$\Psi(\mathcal{F}(1)) \cong \Psi(\mathcal{F})[1] \tag{14}$$

where $\mathcal{F}(1)$ is the twist of $\mathcal{F}$ by the tautological character of $\mathbb{G}_m$;

$$\Psi : \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}[n](n)) \xrightarrow{\sim} \mathrm{Hom}(\Psi(\mathcal{F}), \Psi(\mathcal{G})); \tag{15}$$

$$\langle Im(\Psi) \rangle = D^b(u_q - Mod_0), \tag{16}$$

where $\langle Im(\Psi) \rangle$ denotes the full triangulated subcategory generated by objects of the form $\Psi(\mathcal{F})$, and $u_q - Mod_0$ is the block (direct summand) of the category $u_q - Mod$ which contains the trivial representation;

$$\Psi(\mathcal{O}_{\tilde{\mathcal{N}}}) = \mathbf{k}. \tag{17}$$

The following slight generalization of this result is proved by a straightforward modification of the argument of [BL].

Proposition 8. *Let C be a subtorus in the maximal torus T, and let $u_q - mod^C$ be the category of u_q-modules carrying a compatible grading by weights of C. There exists a functor $\Psi^C : D^b(Coh^{C \times \mathbb{G}_m}(\tilde{\mathcal{N}})) \to D^b(u_q - mod^C)$ satisfying properties* (14)– (17) *above.*

6.3. **Semi-inifinite cohomology as cohomology with support.** From now on we fix C to be a copy of the multiplicative group corresponding to the coweight used to define the grading on u_q (see the beginning of this section), thus we have $u_q - mod^C = u_q - mod$.

Theorem 2. *For $\mathcal{F} \in D^b(Coh^{C \times \mathbb{G}_m})$ we have a canonical isomorphism*

$$Ext_{u_q}^{\infty/2+i}(\mathbf{k}, \Psi(\mathcal{F})) \cong R\Gamma_{\mathfrak{n}}(\pi_*(\mathcal{F})).$$

Proof. We have

$$R\Gamma_{\mathfrak{n}}^{\bullet}(\pi_*(\mathcal{F})) \cong \varinjlim Ext^{\bullet}(\mathcal{O}_{\tilde{\mathcal{N}}}/\pi^*(I), \mathcal{F}),$$

[2] In fact, $D^b(Coh^{\mathbb{G}_m}(\tilde{\mathcal{N}}))$ can be identified with the derived category of a block in the category of graded modules over u_q compatible with a certain grading on u_q, defined in [AJS]. However, unlike the natural grading by weights and its modifications, this grading is neither explicit, nor elementary; it is similar to a grading on the category O of $\mathfrak{g}$ modules with highest weight arising from Hodge weights on the Hom space between Hodge D-modules, or from Frobenius weights.

where I runs over $C \times G_m$ invariant ideals in $\mathcal{O}_{\mathcal{N}}$ with support on $\mathfrak{n}$. We have a canonical arrow $\Psi(\mathcal{O}_{\tilde{\mathcal{N}}}) \to \Psi(\mathcal{O}_{\tilde{\mathcal{N}}}/\pi^*(I))$, and in view of Proposition 7 we have $\Psi(\mathcal{O}_{\tilde{\mathcal{N}}}/\pi^*(I)) \subset D_{\infty/2}$. Thus by Theorem 1 we have a natural map

$$R\Gamma_{\mathfrak{n}}(\pi_*(\mathcal{F})) \longrightarrow Ext^{\infty/2+i}_{u_q}(\mathbf{k}, \Psi(\mathcal{F})).$$

In view of Proposition 1, to check that this map is an isomorphism it suffices to show that the pro-object $\widehat{\Psi(\mathcal{O})}$ in $D_{\infty/2}$ defined by $\widehat{\Psi(\mathcal{O})} = \varprojlim \Psi(\mathcal{O}_{\tilde{\mathcal{N}}}/\pi^*(I))$ represents the same functor on $D_{\infty/2}$ as the object $\mathbf{k} = \Psi(\mathcal{O}) \in D$.

Let $X \in D_{\infty/2}$, and let $f_1, \dots, f_n$ be a regular sequence in $\mathcal{O}(\mathcal{N})$ whose common set of zeroes equals $\mathfrak{n}$. We can and will assume that f_i is an eigen-function for the action of $C \times \mathbb{G}_m$. There exists N such that f_i^N maps to $0 \in End^\bullet(X)$. Then any morphism $\mathbf{k} \to X$ factors through $\mathbf{k}_N = \Psi(\mathcal{O}/(f_i^N))$. This shows that the map $\varinjlim Hom(\mathbf{k}_N, X) \to Hom(\mathbf{k}, X)$ is surjective. Similarly, for large N the map $Hom(\mathbf{k}_N, X) \to Hom(\mathbf{k}_{2N}, X)$ kills the kernel of the map $Hom(\mathbf{k}_N, X) \to Hom(\mathbf{k}, X)$. Thus the map $\varinjlim Hom(\mathbf{k}_N, X) \to Hom(\mathbf{k}, X)$ is injective. $\square$

Corollary 2. *Let T be a tilting module over Lusztig's "big" quantum group U_q. The semi-infinite cohomology $Ext^{\infty/2+i}_{u_q}(\mathbf{k}, T)$ either vanishes or is canonically isomorphic to $R\Gamma_{\mathfrak{n}}(\mathcal{F})$, where $\mathcal{F} \in D^b(Coh^G(\mathcal{N}))$ is a certain (explicit) irreducible object in the heart of the perverse t-structure corresponding to the middle perversity* [B2].

Proof. By the result of [B1] we have $T = \Psi(\tilde{F})$ for some $\tilde{F} \in D^b(Coh^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}}))$, such that $\pi_*(\tilde{F})$ either vanishes or is an (explicit) irreducible perverse equivariant coherent sheaf as above. The statement now follows from Theorem 2.$\square$

Example 3. If $T = \mathbf{k}$ is the trivial module, then it is clear from the construction of [B1] that we can set $\tilde{F} = \mathcal{O}_{\tilde{\mathcal{N}}}$. Thus $\mathcal{F} \cong \mathcal{O}_{\mathcal{N}}$, so Corollary yields the main result of [Ar1].

References

[AJS] H.H. Andersen, J.C. Jantzen, W. Soergel *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p,* Astérisque **220** (1994), 321 pp.

[Ar1] Arkhipov, S., *Semiinfinite cohomology of quantum groups,* Comm. Math. Phys. **188** (1997), no. 2, 379–405.

[Ar2] Arkhipov, S., *Semi-infinite cohomology of associative algebras and bar duality,* Internat. Math. Res. Notices **17** (1997), 833–863.

[Ar3] Arkhipov, S., *Semi-infinite cohomology of quantum groups. II,* in: Topics in quantum groups and finite-type invariants, 3–42, Amer. Math. Soc. Transl. Ser. 2, 185, Amer. Math. Soc., Providence, RI, 1998.

[ABG] Arkhipov, S., Bezrukavnikov, R., Ginzburg V., *Quantum Groups, the loop Grassmannian, and the Springer resolution,* J. Amer. Math. Soc. **17** (2004), 595-678.

[B1] Bezrukavnikov, R., *Cohomology of tilting modules over quantum groups, and t-structures on derived categories of coherent sheaves,* Inv. Math. **166** (2006), no 2, 327–357.

[B2] Bezrukavnikov, R., *Perverse coherent sheaves (after Deligne),* electronic preprint, arXiv:math.AG/0005152.

[B3] Bezrukavnikov, R., *On semi-infinite cohomology of finite dimensional algebras,* electronic preprint, arXiv:math.RT/0005148.

[FS] Bezrukavnikov, R., Finkelberg, M., Schechtman, V., *Factorizable sheaves and quantum groups,* Lecture Notes Math. 1691, Springer Verlag, 1998.

[BL] Bezrukavnikov, R., Lachowska, A., *The small quantum group and the Springer resolution,* Proceedings of Haifa Conference on Quantum Groups (J. Donin memorial volume), Contemporary Math. **433** (2007), 89–101.

[Brz] Brzezinski, T., *The structure of corings: induction functors, Maschke-type theorem, and Frobenius and Galois-type properties*, Algebr. Represent. Theory **5** (2002), #4, p. 389–410.

[BW] Brzezinski, T., Wisbauer, R., *Corings and comodules,* London Mathematical Society Lecture Note Series, 309, Cambridge University Press, Cambridge, 2003.

[GK] Ginzburg, V., Kumar, S., *Cohomology of quantum groups at roots of unity,* Duke Math. J. **69** (1993), no. 1, 179–198.

[H] Hartshorne, R., *Residues and Duality,* Lecture Notes Math. 20, Springer Verlag, 1966.

[L] Lusztig, G., *Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra,* J. Amer. Math. Soc. **3** (1990), no. 1, 257–296.

[P] Positselski, L., *Homological algebra of semimodules and semicontramodules,* electronic preprint, arXiv:0708.3398 [math.CT].

[S] Sevostyanov, A., *Semi-infinite cohomology and Hecke algebras,* Adv. Math. **159** (2001), no. 1, 83–141.

[V] Voronov, A., *Semi-infinite homological algebra,* Invent. Math. **113** (1993), no. 1, 103–146.